**AB Calculus AB Outline**

**Multiple Techniques for Antidifferentiation (Integrals) (Section 4.1)**

- Integration is the inverse of differentiation

For indefinite integrals (integrals where you find all the solutions)

y= $\int f(X)dx = F(X)+C$

**Basic rules**

$\int 0dx = C$

$\int kf(x)dx = kf(x)dx$ + c

$\int kdx = kx + c$

$\int x^n dx = \frac{x^{n-1}}{n+1} + c$ (N $\neq 1$)

- Always put +C at end of every indefinite integral
- Each integrated formula give a family of solutions, this is called a general soultion

Ex:

$\int 5dx = 5x + c$

$\int x^2 dx = \frac{x^3}{3} + c$

$\int \sqrt{x}dx = \frac{2}{3}x^{3/2} + c$

**Definite Integrals are integrals on closed integrals [a,b] (Section 4.3)**

A=Lower Limit
B=Upper limit

- After finding the integral of the function you must evaluate the integral at their endpoints A and B and then find their difference
- +C is disregarded because it cancels itself

Ex: $\int_1^5 x^2 dx = \frac{1}{3}x^3 \; \frac{5^3}{3} - \frac{1^3}{3} = \frac{124}{3}$

**Integration by Substitution (Section 4.5)**

- **used when integrals are too difficult, split integral into U and DU, U is the part that gives the most trouble**
- **U must be the easiest to differentiate**
- **du = the derivative of U**

$\int f(u)du = F(u) + c$

**Ex:** $\int 3(3x-1)^4 dx$ **U=3x-1** **du=3dx**

↑ ↑

du u

***plug back U into the integral***

$\int (u)^4 du = \frac{u^5}{5} + c = \frac{1}{5}(3x-1)^5 + c$

This is integration with U-substitution

**U-substitution (Alternate Method)** **(Only when there is a leftover variable)**

**Ex:** $\int x\sqrt{2x-1}dx$ **u**=2x-1 **x**=$\frac{1}{2}(u+1)$

$\uparrow \quad \uparrow \quad \uparrow$ **du**=2dx **dx**=$\frac{1}{2}du$

$\frac{1}{2}(u+1)$ u $\frac{1}{2}du$

$\int \frac{1}{2}(u+1)\sqrt{u}\frac{1}{2}du = \frac{1}{4}\int (u+1)\sqrt{u}\,du$

$$\frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c$$

(4.4) <u>Fundamental Theorem of Calculus (FTC)</u> (Jessi)

*The Fundamental Theorem of Calculus*

If *f* is continuous on the closed interval [a, b] and *F* is the antiderivative (indefinite integral) of *f* on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This can also be denoted by:

$$\int_a^b f(x)dx = F(x)\Big]_a^b$$

- Using the FTC, the area under a curve can be found by evaluating the integral at the endpoints.

Example:

1) $\int_1^4 (2x-3)^2 dx$

$= \int_1^4 (4x^2 - 6x + 9)dx$

$= \frac{4}{3}x^3 - 3x^2 + 9x\Big]_1^4$

$= \frac{4}{3}(64) - 3(16) + 9(4) - [\frac{4}{3}(1) - 3(1) + 9(1)]$

$= \frac{256}{3} - 48 + 36 - (\frac{22}{3})$

$= 66$

*The Second Fundamental Theorem of Calculus*

If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$, in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

If $f$ is continuous on an open interval $I$ and $a$ any point in $I$, and $F$ is defined by the integral

$$F(x) = \int_a^x f(t)dt$$

then,

$$F'(x) = f(x)$$

- The FTC states that differentiation and integration are inverse operations – they undo each other.

Examples:

1) Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+3}dt\right]$

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+3}dt\right] = \sqrt{x^2+3}$$

2) $F(x) = \int_{\pi/3}^{x^4} sec(t)tan(t)dt$

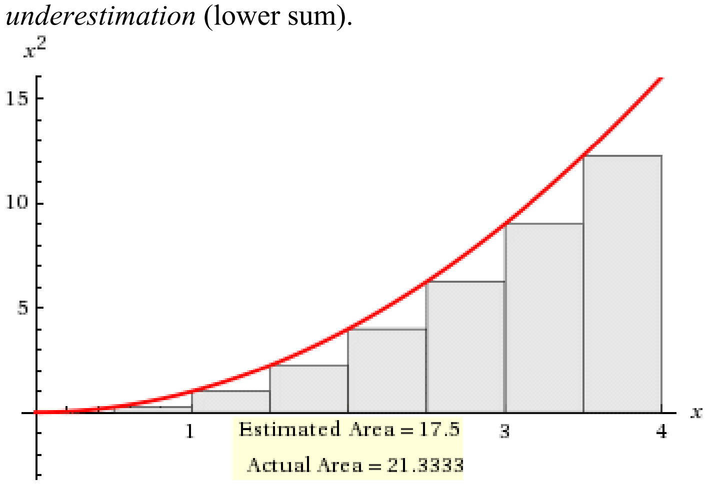
$$F'(x) = 4x^3 sec(x^4)tan(x^4)$$

(4.2 & 4.3) <u>Numerical approximation to definite integrals</u> (Jessi)

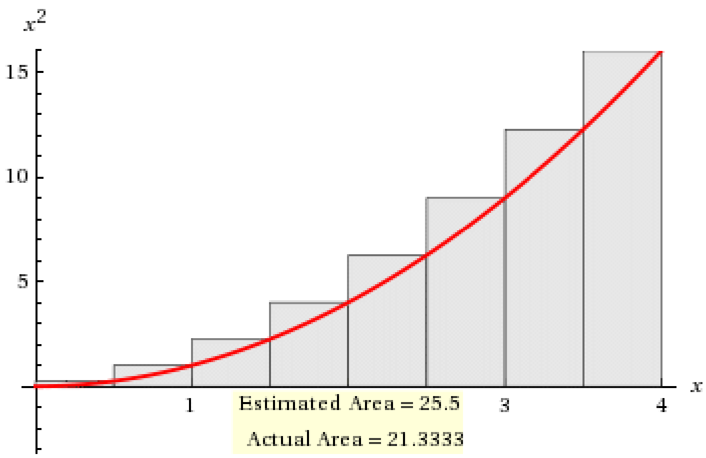
*Riemann sums*

A **Riemann sum** can be used to approximate a definite integral (including definite integrals as the area of a region), by calculating the area of rectangles in the closed interval. There are four types of Riemann sums: **left**, **right**, **middle**, and **trapezoidal**.

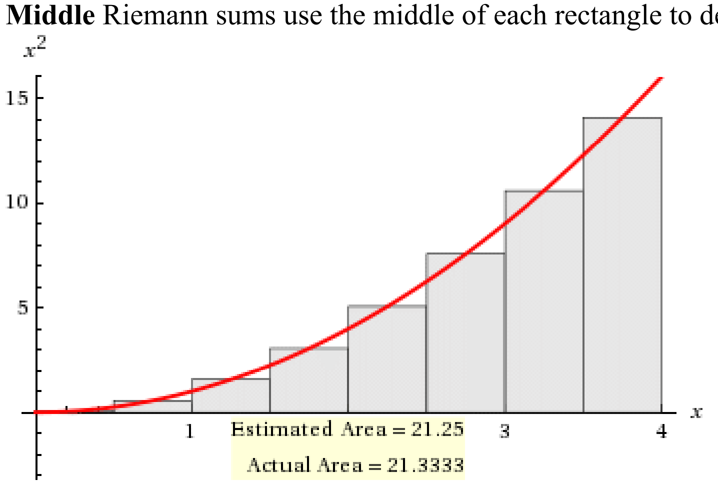
A **left** Riemann sum is calculated when the **back** (or left-endpoint) of each rectangular interval is used to determine the height of the rectangle. In a monotonically *decreasing* function, a left Riemann sum will amount to an *overestimation* (upper sum), and in an *increasing* function, it will amount to an *underestimation* (lower sum).




A **right** Riemann sum is calculated when the **front** (or right-endpoint) is used to determine the height of the rectangle. In decreasing functions, this is the lower sum, and in increasing functions, it is the upper sum.




**Middle** Riemann sums use the middle of each rectangle to determine the height of the rectangle.




For a function *f(x)* continuous over the interval [a, b], where the interval is partitioned so that $a = x_0 < x_1 < x_2 ... < x_n = b$ and the intervals of partition are $\Delta x_1, \Delta x_2 ... \Delta x_n$, the Riemann sum for that function and partition is

$$S = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1})$$

. [ $f(y_i)$ is the height of each rectangle and $(x_i - x_{i-1})$ is the width.]

For all values of *i* in the interval [a, b], the left sum occurs when $y_i = x_{i-1}$, the right sum occurs when $y_i = x_i$, and the middle sum occurs when $y_i = \frac{x_i + x_{i-1}}{2}$. Thus, the area under the curve using *n* rectangles of equal length is approximately:

$$\begin{aligned}\sum_{i=1}^{n} (area\ of\ rectangle_i) &= \sum_{i=1}^{n} f(x_{i-1})\Delta x, left\ endpoint\ rectangles \\ &= \sum_{i=1}^{n} f(x_i)\Delta x, right\ endpoint\ rectangles \\ &= \sum_{i=1}^{n} f(\frac{x_i + x_{i-1}}{2})\Delta x, left\ endpoint\ rectangles\end{aligned}$$

In other words, the area is the base of each rectangle times the height of each rectangle.

(For a left Riemann sum) $A=\left(\frac{b-a}{n}\right)\left(f(x_0)+f(x_1)+\ldots+f(x_{n-1})+f(x_n)\right)$

Example:
1) Use the table of values to find lower and upper estimates of

$$\int_0^{10} f(x)dx$$

Assume that *f* is a decreasing function

| ***x*** | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| ***f(x)*** | 32 | 24 | 12 | - 4 | - 20 | - 36 |

Lower estimate using **right endpoints**:

$$\int_0^{10} f(x)dx=\frac{10-0}{5}(24+12+(-4)+(-20)+(-36))$$
$$=2(-24)$$
$$=-48$$

Upper estimate using **left endpoints**:

$$\int_0^{10} f(x)dx=\frac{10-0}{5}(32+24+12+(-4)+(-20))$$
$$=2(36)$$
$$=88$$

(4.6) **Trapezoidal Rule**
The area under a curve can also be approximated using the areas of trapezoids with heights that are equal in length.

Area of a trapezoid: $\frac{1}{2}h(b_1+b_2)$

If a function is continuous on the interval [a, b], then the height is given by $\frac{b-a}{n}$ (with *n* number of intervals) and the bases are $f(x_0), f(x_1), f(x_2)$...
Using equal intervals, the area can be determined using the general formula:

$$A\approx\frac{b-a}{2n}(f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n))$$

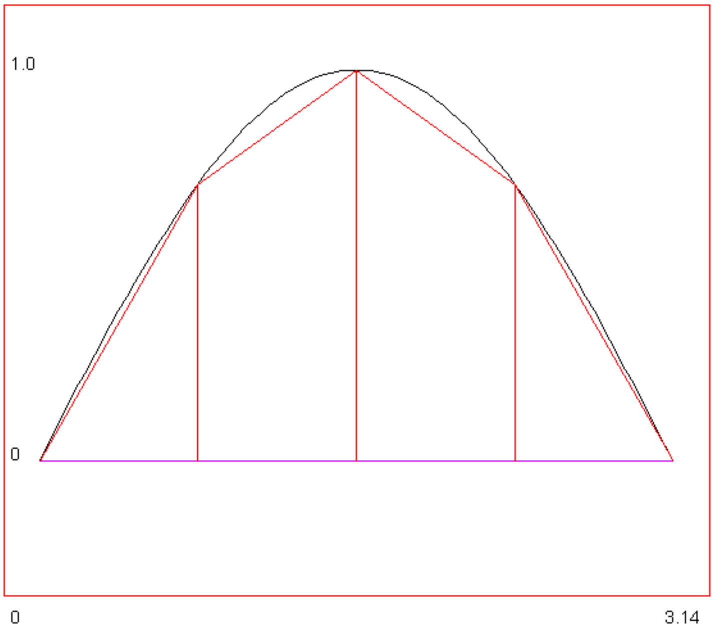
Example:

1) Use the Trapezoid Rule to approximate $\int_0^{\pi} sinxdx$ when n=4

height: $\frac{\pi-0}{4}=\frac{\pi}{4}$

$$\int_0^{\pi} sinxdx\approx\frac{\pi}{2(4)}(f(0)+2f(\tfrac{\pi}{4})+2f(\tfrac{\pi}{2})+2f(\tfrac{3\pi}{4})+f(\pi))$$

$$
\begin{aligned}
&\approx \frac{\pi}{8}(sin0+2sin\frac{\pi}{4}+2sin\frac{\pi}{2}+2sin\frac{3\pi}{4}+sin\pi)\\
&\approx \frac{\pi}{8}(0+\sqrt{2}+2+\sqrt{2}+0)\\
&\approx 1.896
\end{aligned}
$$




Sources:
Weisstein, Eric W. "Second Fundamental Theorem of Calculus." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/SecondFundamentalTheoremofCalculus.html
http://www.cramster.com/definitions/area-using-riemann-sums/169
http://en.wikipedia.org/wiki/Riemann_sum

## Interpretations and Properties of Definite Integrals (4.1, 4.2, 4.4)

A Riemann sum is used to find the area under a curve using rectangles over an interval. The area under the curve is divided up by the rectangles then the area of the rectangles to estimate the area under the curve. A Riemann sum is divided up to a lower and upper sum. The upper sum is found when the front of the rectangle is equal to the height of the curve and the lower found when the back of the rectangle is the height of the curve. Generally the lower Riemann sum is used.

A Riemann sum is not accurate as the sums are above or below the actual area because the rectangle does not fit the curve perfectly. The rectangles though if made small enough will produce an accurate area. Thus the smaller the rectangle is the more accurate the area is. The area under a curve is found by finding the limit as n approaches infinity of the Riemann sum.

This is the same as the Fundamental Theorem of Calculus. The definite integral on the interval [a,b] finds the area under the curve between the endpoints a and b. It can be the definite

integral times f(x) equals the integral of f(b)- integral of f(a) or the definite integral the interval [a,b] of the derivate of f(x) is equal to the f(b) minus f(a).

or

Ex. [1,4]

## Rules for Definite Integrals:

1.

Ex.

The Definite Integral on the interval [a,a] of the function f(x) is equal to 0.

2.

Ex.

The definite integral on the interval [a,b] of the function f(x) plus the definite integral on the interval [b,c] of the function f(x) is equal to the definite integral on the interval [a,c] of the function f(x).

3.

Ex.

The definite integral on the interval [a,b] of the function f(x) is equal to the negative definite integral on the interval [b,a] of the function f(x).

4.

Ex.

The definite integral on the interval [a,b] of a constant times the function f(x) is equal to the constant times the definite integral on the interval [a,b] of the function f(x).

5.

Ex.

The definite integral on the interval [a,b] of the function f(x) plus the function of g(x) is equal to the definite integral on the interval [a,b] of the function f(x) plus the definite integral on the interval [a,b] of the function g(x).